
Presented: 杜文静
E-mail: wjdu@sdu.edu.cn
Telephone: 88399595-2511
Chapter 5 Transient conduction

Contents and objectives

• Time-dependent conditions → unsteady → transient

• Develop procedure for transient conduction → temperature distribution and heat rate

• 0 D transient heat conduction → (lumped capacitance method) 集总热容法 集总参数法

• 1 D transient heat conduction solution for finite and semi-infinite solids (analytical method, 分析解)

• Multi-dimensional heat conduction (numerical calculation)
周期性：物体的温度呈周期性变化；太阳辐射

非周期性：物体的温度不断变化，随时间推移趋近于恒定的值。

基本特征：在热量传递方向上，不同位置处的导热流量是不相同的。

热阻方法一般不适用于非稳态导热问题。
Brief Introduction

Three phases for transient conduction

1. non-regular regime: 非正规状况阶段  temp distribution is mainly determined by initial temp;

2. regular regime: 正规状况阶段  temp distribution is mainly determined by boundary conditions;

3. new steady state: 新的稳态
Discussion on Biot number

Definition on Biot number $\text{Bi}$ (with the 3rd boundary condition)

$$R_{\text{conduction}} = \frac{L}{kA}$$
$$R_{\text{convection}} = \frac{1}{hA}$$

$$\text{Bi} = \frac{R_{\text{conduction}}}{R_{\text{convection}}} = \frac{L/k}{1/h} = \frac{Lh}{k}$$

Biot number, 毕渥数
无量纲参数, 准则数,
特征数 characteristic number
特征长度, characteristic length

Fundamentals of Heat and Mass Transfer, 6th
Relation between Biot number and temp distribution

1. $\text{Bi} \rightarrow 0 \quad K \rightarrow \infty$ internal temperature distribution is uniform.

2. If $0 < \text{Bi} < \infty$, temperature different between surface and surroundings & non-uniform temp distribution inside the body.

3. $\text{Bi} \rightarrow \infty \quad h \rightarrow \infty$ surface temp is dropped to $T_\infty$.
5.1 the lumped capacitance method

For Case 1: lumped capacitance method

Essence: the assumption that the temperature of the solid is spatially uniform at any instant during the transient process

- Temp gradient within the solid is negligible
- Thermal conductivity is infinitely large
- Or size is infinitely small
- Or convection heat transfer coefficient is infinitely small

\[ \dot{E}_{\text{out}} = \dot{E}_{\text{st}} \]

\[-hA_s(T - T_\infty) = \rho Vc \frac{dT}{dt} \]

\[ \theta \equiv T - T_\infty \]

\[ \frac{\rho Vc}{hA_s} \frac{d\theta}{dt} = -\theta \]

\[ \frac{\rho Vc}{hA_s} \int_{\theta_i}^{\theta} \frac{d\theta}{\theta} = -\int_0^t dt \]

\[ \frac{\rho Vc}{hA_s} \ln \frac{\theta_i}{\theta} = t \]

\[ \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp \left[ - \left( \frac{hA_s}{\rho Vc} \right) t \right] \]
5.1 the lumped capacitance method

\[ \frac{\theta}{\theta_i} = \frac{T - T_{\infty}}{T_i - T_{\infty}} = \exp \left[ -\left( \frac{hA_s}{\rho Vc} \right) t \right] \]

Time required for a solid to reach a temp

\[ t = \frac{\rho Vc}{hA_s} \ln \frac{\theta_i}{\theta} \]

Thermal time constant \( \tau_t \)

Convection thermal resistance, \( R_t \)

Lumped thermal capacitance, \( C_t \)

Heat transfer during a time period

\[ Q = \int_0^t q \, dt = hA_s \int_0^t \theta \, dt \]

\[ Q = (\rho Vc) \theta_i \left[ 1 - \exp\left(-\frac{t}{\tau_t}\right) \right] \]
5.1 the lumped capacitance method

- Transient temp response for different thermal time constant

\[ \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{t}{\tau_t}\right) \]

**Example:**

Thermocouple

例如  \( t = \tau_t \)

\[ \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left(-\frac{t}{\tau_t}\right) = \exp(-1) = 0.368 \]
5.2 validity of the lumped capacitance method

Based on the surface energy balance

\[
\frac{kA}{L} (T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)
\]

\[
\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{hL}{k} = Bi
\]

- \(h\) \(\rightarrow\) convection or radiation coefficient
- \(k\) \(\rightarrow\) thermal conductivity of the solid
- \(L_c\) \(\rightarrow\) characteristic length of the solid (\(\sqrt{\frac{k}{hA}}\))

\[
L_c = \begin{cases} 
L, & \text{for a plane wall of thickness } 2L \\
r/2, & \text{for a long cylinder} \\
r/3, & \text{for a sphere}
\end{cases}
\]

For the 3rd boundary condition

Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.
5.2 validity of the lumped capacitance method

- Criterion for applicability of the Lumped Capacitance Method:

\[
Bi = \frac{h L_c}{k} < 0.1
\]

\[
\frac{hA_s t}{\rho Vc} = \frac{ht}{\rho c L_c} = \frac{h L_c}{k} \cdot \frac{k}{\rho c} \frac{t}{L_c^2} = Bi \cdot Fo
\]

\[
\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)
\]

The very first thing when confronted with a transient problem is to calculate the Biot number.

Biot number of thermocouple is around 0.001 or even smaller and so it can provide an accurate temperature measurement value.
5.2 validity of the lumped capacitance method

- Physical interpretation of Fourier number $F_l$ (dimensionless time)

$$q_{con} \sim \frac{kL^2 \Delta T}{L} \quad \dot{E}_{st} \sim \rho L^3 c \Delta T / t$$

$$F_l = \frac{\alpha t}{L^2} = \frac{kt}{\rho c L^2} \sim \frac{q}{\dot{E}_{st}} = \frac{\text{condution heat rate}}{\text{energystorage rate}}$$

Fourier number provides a measure of the relative effectiveness with which a solid conducts and stores thermal energy.

- Bi: internal thermal resistance of a solid to the boundary layer thermal resistance 固体内部的导热热阻和边界上的换热热阻之比
**Example 5.1**

A thermocouple junction, which may be approximated as a sphere, is to be used for temperature measurement in a gas stream. The convection coefficient between the junction surface and the gas is \( h = 400 \text{ W/m}^2 \cdot \text{K} \), and the junction thermophysical properties are \( k = 20 \text{ W/m} \cdot \text{K} \), \( c = 400 \text{ J/kg} \cdot \text{K} \), and \( \rho = 8500 \text{ kg/m}^3 \). Determine the junction diameter needed for the thermocouple to have a time constant of 1 s. If the junction is at 25°C and is placed in a gas stream that is at 200°C, how long will it take for the junction to reach 199°C?

**Schematic:**

\[
T_\infty = 200°C \\
h = 400 \text{ W/m}^2 \cdot \text{K}
\]

\[
\text{Gas stream}
\]

\[
D
\]

\[
\text{Thermocouple junction} \\
T_i = 25°C
\]

\[
k = 20 \text{ W/m} \cdot \text{K} \\
c = 400 \text{ J/kg} \cdot \text{K} \\
\rho = 8500 \text{ kg/m}^3
\]
厨师吹肉丝

一厨师在炒鸡肉丝时要品尝一下咸淡，于是他从100℃的热炒锅中取出一鸡肉丝，用口吹了一会，待其降至65℃时再放入口中。试估算厨师需要吹多长时间？出锅时鸡肉丝可视为平均直径为2mm的圆条，厨师口中吹出的气流温度为30℃，其与鸡肉丝之间的表面传热系数为100W/(m²·K)，鸡肉丝的
\[ \rho = 810\text{kg/m}^3, \quad c = 3.35\text{kJ/(kg·K)}, \quad \lambda = 1.1\text{W/(m·K)}. \]

解：首先检验是否可用集总参数法。为此计算 Biₐ

\[ Bi_v = \frac{h(V/A)}{\lambda} = \frac{h(\pi r^2 l)}{2\pi rl} = \frac{hr}{2\lambda} \]
\[ = \frac{100\text{W/(m}^2\cdot\text{K)×1×10}^3\text{m}}{2\times1.1\text{W/(m}^2\cdot\text{K)}} \]
\[ = 0.045 < 0.05 \]

故可以采用集总参数法。

又

\[ \frac{hA}{c\rho V} = \frac{2h}{c\rho r} = \frac{2\times100\text{W/(m}^2\cdot\text{K)}}{3.5\times10^3\text{J/(kg·K)}\times810\text{kg/m}^3} \]
\[ = 0.0705\text{s}^{-1} \]

\[ \frac{\theta}{\theta_0} = \frac{t-t_\infty}{t_0-t_\infty} = \frac{65\degree\text{C} - 100\degree\text{C}}{30\degree\text{C} - 100\degree\text{C}} = 0.5 \]

所以
\[ 0.5 = \exp(-0.0705\tau) \]

由此解得：\[ \tau = 9.83\text{s} \]
5.3 general lumped capacitance analysis

General conditions analysis:

- Convection
- Radiation
- Internal energy generation
- Apply a Surface heat flux
- Volumetric or sur

\[ q_s A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{\text{sur}}^4)]A_{s(c,r)} = \rho Vc \frac{dT}{dt} \]

No exact solution for this general conditions.

**Exact solutions exist obtain with simplified condition.**
5.3 general lumped capacitance analysis

- Exact solutions exist with 2 simplified conditions
  1) Only radiation

\[
\rho V c \frac{dT}{ds} = \sigma(T^4 - T_{\text{sur}}^4)
\]

\[
e = \frac{\rho V c}{4 \varepsilon A_{s,r} \sigma T_{\text{sur}}^3} \ln \left| \frac{T_{\text{sur}} + T}{T_{\text{sur}} - T} \right| - \ln \left| \frac{T_{\text{sur}} + T_i}{T_{\text{sur}} - T_i} \right|
\]

\[
+ 2 \left[ \tan^{-1} \left( \frac{T}{T_{\text{sur}}} \right) - \tan^{-1} \left( \frac{T_i}{T_{\text{sur}}} \right) \right]
\]

\[
t = \frac{\rho V c}{3 \varepsilon A_{s,r} \sigma} \left( \frac{1}{T^3} - \frac{1}{T_i^3} \right)
\]

If \( T_{\text{sur}} = 0 \) K, deep space condition

study by yourself
5.3 general lumped capacitance analysis

- Exact solutions exist with 2 simplified conditions
  2) \( \rightarrow \) No radiation & \( h \) is independent of time

\[
q'' A = \left[ 1 - \varepsilon \sigma (T^4 - T_{\text{surf}}^4)A_{s(c,r)} \right] = \rho Vc \frac{dT}{dt} \]

\[
\frac{d\theta}{dt} + a\theta - b = 0
\]

\[
a \equiv \frac{(hA_{s,c} / \rho Vc)}{b \equiv [(q''A_{s,h} + E_g) / \rho Vc]}
\]

\[
\frac{T - T_\infty}{T_i - T_\infty} = \exp(-at) + \frac{b/a}{T_i - T_\infty} [1 - \exp(-at)]
\]

Example 5.2 p265

Discussion

\[
b = 0, \quad t \rightarrow \infty (T - T_\infty) = (b/a),
\]
Problem 5.12: Charging a thermal energy storage system consisting of a packed bed of aluminum spheres.

KNOWN: Diameter, density, specific heat and thermal conductivity of aluminum spheres used in packed bed thermal energy storage system. Convection coefficient and inlet gas temperature.

FIND: Time required for sphere at inlet to acquire 90% of maximum possible thermal energy and the corresponding center temperature.

Schematic:

Gas
$T_{g,i}$, $h$

$T_{g,i} = 300^\circ$C
$h = 75$ W/m$^2$-K

Aluminum sphere
$D = 75$ mm, $T_i = 25^\circ$C
$\rho = 2700$ kg/m$^3$
$c = 950$ J/kg-K
$k = 240$ W/m-K
ASSUMPTIONS: (1) Negligible heat transfer to or from a sphere by radiation or conduction due to contact with other spheres, (2) Constant properties.

ANALYSIS: To determine whether a lumped capacitance analysis can be used, first compute

\[ Bi = \frac{h(r_0/3)}{k} = \frac{75 \text{ W/m}^2\cdot\text{K}}{150 \text{ W/m}\cdot\text{K}} = 0.013 \ll 1. \]

Hence, the lumped capacitance approximation may be made, and a uniform temperature may be assumed to exist in the sphere at any time.

From Eq. 5.8a, achievement of **90% of the maximum possible thermal energy storage** corresponds to

\[ \frac{\Delta E_{st}}{\rho c V \theta_1} = 0.90 = 1 - \exp\left(-\frac{t}{\tau_t}\right) \]

\[ \tau_t = \frac{\rho V c}{h A_s} = \frac{\rho D c}{6h} = \frac{2700 \text{ kg/m}^3 \times 0.075 \text{ m} \times 950 \text{ J/kg} \cdot \text{K}}{6 \times 75 \text{ W/m}^2 \cdot \text{K}} = 427 \text{s}. \]

\[ t = -\tau_t \ln(0.1) = 427 \times 2.30 = 984 \text{s} \]

From Eq. (5.6), the corresponding temperature at any location in the sphere is

\[ T(984\text{s}) = T_{g,i} + \left(T_i - T_{g,i}\right) \exp\left(-6ht / \rho D c\right) \]

\[ T(984\text{s}) = 300^\circ \text{C} - 275^\circ \text{C} \exp\left(-6 \times 75 \text{ W/m}^2 \cdot \text{K} \times 984 \text{s} / 2700 \text{ kg/m}^3 \times 0.075 \text{ m} \times 950 \text{ J/kg} \cdot \text{K}\right) \]

\[ T(984\text{s}) = 272.5^\circ \text{C} \]

If the product of the density and specific heat of copper is \((\rho c)_\text{Cu} \approx 8900 \text{ kg/m}^3 \times 400 \text{ J/kg} \cdot \text{K} = 3.56 \times 10^6 \text{ J/m}^3 \cdot \text{K}, is there any advantage to using copper spheres of equivalent diameter in lieu of aluminum spheres?
5.4 spatial effects

- Temp gradients within media are no longer negligible.
- For 1 D case

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

\[ T(x, 0) = T_i \]

\[ \frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \]

\[ -k \frac{\partial T}{\partial x} \bigg|_{x=L} = h[T(L, t) - T_\infty] \]

\[ T = T(x, t, T_i, T_\infty, L, k, \alpha, h) \]

Non-dimensionalizing governing equations

$$\frac{\partial^2 \theta^*}{\partial x^*^2} = \frac{\partial \theta^*}{\partial F.o}$$

\[ \theta^*(x^*, 0) = 1 \]

\[ \frac{\partial \theta^*}{\partial x^*} \bigg|_{x^*=0} = 0 \]

\[ \frac{\partial \theta^*}{\partial x^*} \bigg|_{x^*=1} = -Bi \theta^*(1, t^*) \]

$$\theta^* = f(x^*, F.o, Bi)$$
5.4 spatial effects

• Non-dimensionalizing governing equations

→ Arrange relevant variables into suitable groups

\[ \theta^* \equiv \frac{\theta - T_\infty}{T_i - T_\infty} \]

Dimensionless temperature difference

\[ x^* \equiv \frac{x}{L} \]

Dimensionless spatial coordinate

\[ t^* \equiv \frac{\alpha t}{L^2} \equiv Fo \]

Dimensionless time
5.5 the plane wall with convection

- Exact solution

\[
\theta^* = \sum_{n=1}^{\infty} C_n \exp \left(-\zeta_n^2Fo\right) \cos \left(\zeta_n x^*\right)
\]

\[
C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin (2\zeta_n)}
\]

\[
\zeta_n \tan \zeta_n = Bi \quad n = 1,2,3,\ldots
\]
### B.3

The First Four Roots of the Transcendental Equation, $\xi_n \tan \xi_n = Bi$, for Transient Conduction in a Plane Wall

<table>
<thead>
<tr>
<th>$Bi = \frac{hL}{k}$</th>
<th>$\xi_1$</th>
<th>$\xi_2$</th>
<th>$\xi_3$</th>
<th>$\xi_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3.1416</td>
<td>6.2832</td>
<td>9.4248</td>
</tr>
<tr>
<td>0.001</td>
<td>0.0316</td>
<td>3.1419</td>
<td>6.2833</td>
<td>9.4249</td>
</tr>
<tr>
<td>0.002</td>
<td>0.0447</td>
<td>3.1422</td>
<td>6.2835</td>
<td>9.4250</td>
</tr>
<tr>
<td>0.004</td>
<td>0.0632</td>
<td>3.1429</td>
<td>6.2838</td>
<td>9.4252</td>
</tr>
<tr>
<td>0.006</td>
<td>0.0774</td>
<td>3.1435</td>
<td>6.2841</td>
<td>9.4254</td>
</tr>
<tr>
<td>0.008</td>
<td>0.0893</td>
<td>3.1441</td>
<td>6.2845</td>
<td>9.4256</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0998</td>
<td>3.1448</td>
<td>6.2848</td>
<td>9.4258</td>
</tr>
<tr>
<td>0.02</td>
<td>0.1410</td>
<td>3.1479</td>
<td>6.2864</td>
<td>9.4269</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1987</td>
<td>3.1543</td>
<td>6.2895</td>
<td>9.4290</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2425</td>
<td>3.1606</td>
<td>6.2927</td>
<td>9.4311</td>
</tr>
<tr>
<td>0.08</td>
<td>0.2791</td>
<td>3.1668</td>
<td>6.2959</td>
<td>9.4333</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3111</td>
<td>3.1731</td>
<td>6.2991</td>
<td>9.4354</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4328</td>
<td>3.2039</td>
<td>6.3148</td>
<td>9.4459</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5218</td>
<td>3.2341</td>
<td>6.3305</td>
<td>9.4565</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5932</td>
<td>3.2636</td>
<td>6.3461</td>
<td>9.4670</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6533</td>
<td>3.2923</td>
<td>6.3616</td>
<td>9.4775</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7051</td>
<td>3.3204</td>
<td>6.3770</td>
<td>9.4879</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7506</td>
<td>3.3477</td>
<td>6.3923</td>
<td>9.4983</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7910</td>
<td>3.3744</td>
<td>6.4074</td>
<td>9.5087</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8274</td>
<td>3.4003</td>
<td>6.4224</td>
<td>9.5190</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8603</td>
<td>3.4256</td>
<td>6.4373</td>
<td>9.5293</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9882</td>
<td>3.5422</td>
<td>6.5097</td>
<td>9.5801</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0709</td>
<td>3.6436</td>
<td>6.5783</td>
<td>9.6296</td>
</tr>
<tr>
<td>3.0</td>
<td>1.1925</td>
<td>3.8088</td>
<td>6.7040</td>
<td>9.7240</td>
</tr>
<tr>
<td>4.0</td>
<td>1.2646</td>
<td>3.9352</td>
<td>6.8140</td>
<td>9.8119</td>
</tr>
<tr>
<td>5.0</td>
<td>1.3138</td>
<td>4.0336</td>
<td>6.9096</td>
<td>9.8928</td>
</tr>
<tr>
<td>6.0</td>
<td>1.3496</td>
<td>4.1116</td>
<td>6.9924</td>
<td>9.9667</td>
</tr>
<tr>
<td>7.0</td>
<td>1.3766</td>
<td>4.1746</td>
<td>7.0640</td>
<td>10.0339</td>
</tr>
<tr>
<td>8.0</td>
<td>1.3978</td>
<td>4.2264</td>
<td>7.1263</td>
<td>10.0949</td>
</tr>
<tr>
<td>9.0</td>
<td>1.4149</td>
<td>4.2694</td>
<td>7.1806</td>
<td>10.1502</td>
</tr>
<tr>
<td>10.0</td>
<td>1.4289</td>
<td>4.3058</td>
<td>7.2281</td>
<td>10.2003</td>
</tr>
<tr>
<td>15.0</td>
<td>1.4729</td>
<td>4.4255</td>
<td>7.3959</td>
<td>10.3898</td>
</tr>
<tr>
<td>20.0</td>
<td>1.4961</td>
<td>4.4915</td>
<td>7.4954</td>
<td>10.5117</td>
</tr>
<tr>
<td>30.0</td>
<td>1.5202</td>
<td>4.5615</td>
<td>7.6057</td>
<td>10.6543</td>
</tr>
<tr>
<td>40.0</td>
<td>1.5325</td>
<td>4.5979</td>
<td>7.6647</td>
<td>10.7334</td>
</tr>
<tr>
<td>50.0</td>
<td>1.5400</td>
<td>4.6202</td>
<td>7.7012</td>
<td>10.7832</td>
</tr>
<tr>
<td>60.0</td>
<td>1.5451</td>
<td>4.6353</td>
<td>7.7259</td>
<td>10.8172</td>
</tr>
<tr>
<td>80.0</td>
<td>1.5514</td>
<td>4.6543</td>
<td>7.7573</td>
<td>10.8606</td>
</tr>
<tr>
<td>100.0</td>
<td>1.5552</td>
<td>4.6658</td>
<td>7.7764</td>
<td>10.8871</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.5708</td>
<td>4.7124</td>
<td>7.8540</td>
<td>10.9956</td>
</tr>
</tbody>
</table>

The first four root in Appendix B.3
5.5 the plane wall with convection

- One term approximation (近似解)， if $Fo > 0.2$ then error $< 1\%$

\[
\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*)
\]

\[
\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*)
\]

The midplane temp distribution

\[
\theta_o^* = C_1 \exp(-\zeta_1^2 Fo)
\]

\[
\theta^* = \theta_o^* \cos(\zeta_1 x^*)
\]
5.5 the plane wall with convection

• Total energy transfer
  (for the calculation of the total energy left or enter the wall up to any time
  in the transient process)

From initial state to ultimate steady state time\(=\infty\)

The maximum energy transfer

\[
Q_o = \rho c V (T_i - T_\infty)
\]

\[
\frac{Q}{Q_o} = 1 - \frac{\sin \zeta_1}{\zeta_1} \theta_o^*
\]

where

\[
\theta_o^* = C_1 \exp (- \zeta_1^2 Fo)
\]
5.6 radial system with convection

- Exact solution

Infinite cylinder \((L/r_0>=10)\) sphere

\[
\theta^* = \sum_{n=1}^{\infty} C_n \exp \left(-\zeta_n^2 Fo\right) J_0(\zeta_n r^*)
\]

\[
C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0(\zeta_n)}
\]

\[
\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi
\]

J Bessel function
### B.4

**Bessel Functions of the First Kind**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$J_0(x)$</th>
<th>$J_1(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9975</td>
<td>0.0499</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9900</td>
<td>0.0995</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9776</td>
<td>0.1483</td>
</tr>
<tr>
<td>0.4</td>
<td>0.9604</td>
<td>0.1960</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9385</td>
<td>0.2423</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9120</td>
<td>0.2867</td>
</tr>
<tr>
<td>0.7</td>
<td>0.8812</td>
<td>0.3290</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8463</td>
<td>0.3688</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8075</td>
<td>0.4059</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7652</td>
<td>0.4400</td>
</tr>
<tr>
<td>1.1</td>
<td>0.7196</td>
<td>0.4709</td>
</tr>
<tr>
<td>1.2</td>
<td>0.6711</td>
<td>0.4983</td>
</tr>
<tr>
<td>1.3</td>
<td>0.6201</td>
<td>0.5220</td>
</tr>
<tr>
<td>1.4</td>
<td>0.5669</td>
<td>0.5419</td>
</tr>
<tr>
<td>1.5</td>
<td>0.5118</td>
<td>0.5579</td>
</tr>
<tr>
<td>1.6</td>
<td>0.4554</td>
<td>0.5699</td>
</tr>
<tr>
<td>1.7</td>
<td>0.3980</td>
<td>0.5778</td>
</tr>
<tr>
<td>1.8</td>
<td>0.3400</td>
<td>0.5815</td>
</tr>
<tr>
<td>1.9</td>
<td>0.2818</td>
<td>0.5812</td>
</tr>
<tr>
<td>2.0</td>
<td>0.2239</td>
<td>0.5767</td>
</tr>
<tr>
<td>2.1</td>
<td>0.1666</td>
<td>0.5683</td>
</tr>
<tr>
<td>2.2</td>
<td>0.1104</td>
<td>0.5560</td>
</tr>
<tr>
<td>2.3</td>
<td>0.0555</td>
<td>0.5399</td>
</tr>
<tr>
<td>2.4</td>
<td>0.0025</td>
<td>0.5202</td>
</tr>
</tbody>
</table>
5.6 radial system with convection

- **Approximate solutions** (one term approximation, \( F_o > 0.2 \))

### Infinite cylinder

\[
\theta^* = C_1 \exp \left( -\xi_1^2 F_o \right) J_0 (\xi_1 r^*)
\]

\[
\theta^* = \theta_o^* J_0 (\xi_1 r^*)
\]

\[
\theta_o^* = C_1 \exp \left( -\xi_1^2 F_o \right)
\]

### Sphere

\[
\theta^* = C_1 \exp \left( -\xi_1^2 F_o \right) \frac{1}{\xi_1 r^* \sin (\xi_1 r^*)}
\]

\[
\theta^* = \theta_o^* \frac{1}{\xi_1 r^* \sin (\xi_1 r^*)}
\]

\[
\theta_o^* = C_1 \exp \left( -\xi_1^2 F_o \right)
\]

- **Total energy transfer**

\[
\frac{Q}{Q_o} = 1 - 2 \theta_o^* \frac{1}{\xi_1} J_1 (\xi_1)
\]

\[
\frac{Q}{Q_o} = 1 - \frac{3 \theta_o^*}{\xi_1^3} \left[ \sin (\xi_1) - \xi_1 \cos (\xi_1) \right]
\]
<table>
<thead>
<tr>
<th>$Bi^3$</th>
<th>Plane Wall</th>
<th></th>
<th>Infinite Cylinder</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\zeta_1$</td>
<td>$C_1$</td>
<td>$\zeta_1$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0998</td>
<td>1.0017</td>
<td>0.1412</td>
<td>1.0025</td>
</tr>
<tr>
<td>0.02</td>
<td>0.1410</td>
<td>1.0033</td>
<td>0.1995</td>
<td>1.0050</td>
</tr>
<tr>
<td>0.03</td>
<td>0.1723</td>
<td>1.0049</td>
<td>0.2440</td>
<td>1.0075</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1987</td>
<td>1.0066</td>
<td>0.2814</td>
<td>1.0099</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2218</td>
<td>1.0082</td>
<td>0.3143</td>
<td>1.0124</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2425</td>
<td>1.0098</td>
<td>0.3438</td>
<td>1.0148</td>
</tr>
<tr>
<td>0.07</td>
<td>0.2615</td>
<td>1.0114</td>
<td>0.3709</td>
<td>1.0173</td>
</tr>
<tr>
<td>0.08</td>
<td>0.2791</td>
<td>1.0130</td>
<td>0.3960</td>
<td>1.0197</td>
</tr>
<tr>
<td>0.09</td>
<td>0.2956</td>
<td>1.0145</td>
<td>0.4195</td>
<td>1.0222</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3111</td>
<td>1.0161</td>
<td>0.4417</td>
<td>1.0246</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3779</td>
<td>1.0237</td>
<td>0.5376</td>
<td>1.0365</td>
</tr>
<tr>
<td>0.20</td>
<td>0.4328</td>
<td>1.0311</td>
<td>0.6170</td>
<td>1.0483</td>
</tr>
<tr>
<td>0.25</td>
<td>0.4801</td>
<td>1.0382</td>
<td>0.6856</td>
<td>1.0598</td>
</tr>
<tr>
<td>0.30</td>
<td>0.5218</td>
<td>1.0450</td>
<td>0.7465</td>
<td>1.0712</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5932</td>
<td>1.0580</td>
<td>0.8516</td>
<td>1.0932</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6533</td>
<td>1.0701</td>
<td>0.9408</td>
<td>1.1143</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7051</td>
<td>1.0814</td>
<td>1.0184</td>
<td>1.1345</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7506</td>
<td>1.0919</td>
<td>1.0873</td>
<td>1.1539</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7910</td>
<td>1.1016</td>
<td>1.1490</td>
<td>1.1724</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8274</td>
<td>1.1107</td>
<td>1.2048</td>
<td>1.1902</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8603</td>
<td>1.1191</td>
<td>1.2558</td>
<td>1.2071</td>
</tr>
<tr>
<td>2.0</td>
<td>1.0769</td>
<td>1.1785</td>
<td>1.5994</td>
<td>1.3384</td>
</tr>
<tr>
<td>3.0</td>
<td>1.1925</td>
<td>1.2102</td>
<td>1.7887</td>
<td>1.4191</td>
</tr>
<tr>
<td>4.0</td>
<td>1.2646</td>
<td>1.2287</td>
<td>1.9081</td>
<td>1.4698</td>
</tr>
<tr>
<td>5.0</td>
<td>1.3138</td>
<td>1.2402</td>
<td>1.9898</td>
<td>1.5029</td>
</tr>
<tr>
<td>6.0</td>
<td>1.3496</td>
<td>1.2479</td>
<td>2.0490</td>
<td>1.5253</td>
</tr>
<tr>
<td>7.0</td>
<td>1.3766</td>
<td>1.2532</td>
<td>2.0937</td>
<td>1.5411</td>
</tr>
<tr>
<td>8.0</td>
<td>1.3978</td>
<td>1.2570</td>
<td>2.1286</td>
<td>1.5526</td>
</tr>
<tr>
<td>9.0</td>
<td>1.4149</td>
<td>1.2598</td>
<td>2.1566</td>
<td>1.5611</td>
</tr>
<tr>
<td>10.0</td>
<td>1.4289</td>
<td>1.2620</td>
<td>2.1795</td>
<td>1.5677</td>
</tr>
<tr>
<td>20.0</td>
<td>1.4961</td>
<td>1.2699</td>
<td>2.2881</td>
<td>1.5919</td>
</tr>
<tr>
<td>30.0</td>
<td>1.5202</td>
<td>1.2717</td>
<td>2.3261</td>
<td>1.5973</td>
</tr>
<tr>
<td>40.0</td>
<td>1.5325</td>
<td>1.2723</td>
<td>2.3455</td>
<td>1.5993</td>
</tr>
<tr>
<td>50.0</td>
<td>1.5400</td>
<td>1.2727</td>
<td>2.3572</td>
<td>1.6002</td>
</tr>
<tr>
<td>100.0</td>
<td>1.5552</td>
<td>1.2731</td>
<td>2.3899</td>
<td>1.6015</td>
</tr>
</tbody>
</table>
5S.1
Graphical Representation of One-Dimensional, Transient Conduction in the Plane Wall, Long Cylinder, and Sphere

- Heisler chart 海斯勒图 or nomogram 诺膜图

**Figure 5S.1** Midplane temperature as a function of time for a plane wall of thickness 2L [1]. Used with permission.
5S.1

Graphical Representation of One-Dimensional, Transient Conduction in the Plane Wall, Long Cylinder, and Sphere

**Figure 5S.2**  Temperature distribution in a plane wall of thickness $2L$ [1]. Used with permission.
5S.1
Graphical Representation of One-Dimensional, Transient Conduction in the Plane Wall, Long Cylinder, and Sphere

Figure 5S.3 Internal energy change as a function of time for a plane wall of thickness $2L$ [2]. Adapted with permission.
5.6 radial system with convection

- Example 5.4  p278, plane wall
  (one term approximation, 单项近似)

- Example 5.5  p280, sphere
5.7 semi-infinite solid

- **semi-infinite solid** (半无限大) : a solid which can extend in all directions but in one direction with an identifiable surface.

\[ T(x \to \infty, t) = T_i \]

- Analytical solutions in three boundary conditions

![Diagram showing transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.](image-url)
5.7 semi-infinite solid

- **Case 1 constant surface temp** $T(0,t) = T_s$

**A similarity variable** 相似变量

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

Transform the partial differential equation into an ordinary differential one

$$\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}$$

**Gaussian error function** 高斯误差函数

$$\frac{T - T_s}{T_i - T_s} = \frac{1}{\sqrt{\pi} \sqrt{\alpha t}} \int_0^\eta \exp(-u^2) \, du \equiv \text{erf} \, \eta$$

Heat flux through semi-infinite solid

$$q''_s(t) = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

$\rightarrow$ Appendix B
### B.2 Gaussian Error Function

The Gaussian error function is defined as

\[ \text{erf } w = \frac{2}{\sqrt{\pi}} \int_{0}^{w} e^{-v^2} \, dv \]

The complementary error function is defined as

\[ \text{erfc } w = 1 - \text{erf } w \]

<table>
<thead>
<tr>
<th>( w )</th>
<th>( \text{erf } w )</th>
<th>( w )</th>
<th>( \text{erf } w )</th>
<th>( w )</th>
<th>( \text{erf } w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00000</td>
<td>0.36</td>
<td>0.38933</td>
<td>1.04</td>
<td>0.85865</td>
</tr>
<tr>
<td>0.02</td>
<td>0.02256</td>
<td>0.38</td>
<td>0.40901</td>
<td>1.08</td>
<td>0.87333</td>
</tr>
<tr>
<td>0.04</td>
<td>0.04511</td>
<td>0.40</td>
<td>0.42839</td>
<td>1.12</td>
<td>0.88679</td>
</tr>
<tr>
<td>0.06</td>
<td>0.06762</td>
<td>0.44</td>
<td>0.46622</td>
<td>1.16</td>
<td>0.89910</td>
</tr>
<tr>
<td>0.08</td>
<td>0.09008</td>
<td>0.48</td>
<td>0.50275</td>
<td>1.20</td>
<td>0.91031</td>
</tr>
<tr>
<td>0.10</td>
<td>0.11246</td>
<td>0.52</td>
<td>0.53790</td>
<td>1.30</td>
<td>0.93401</td>
</tr>
<tr>
<td>0.12</td>
<td>0.13476</td>
<td>0.56</td>
<td>0.57162</td>
<td>1.40</td>
<td>0.95228</td>
</tr>
<tr>
<td>0.14</td>
<td>0.15695</td>
<td>0.60</td>
<td>0.60386</td>
<td>1.50</td>
<td>0.96611</td>
</tr>
<tr>
<td>0.16</td>
<td>0.17901</td>
<td>0.64</td>
<td>0.63459</td>
<td>1.60</td>
<td>0.97635</td>
</tr>
<tr>
<td>0.18</td>
<td>0.20094</td>
<td>0.68</td>
<td>0.66378</td>
<td>1.70</td>
<td>0.98379</td>
</tr>
<tr>
<td>0.20</td>
<td>0.22270</td>
<td>0.72</td>
<td>0.69143</td>
<td>1.80</td>
<td>0.98909</td>
</tr>
<tr>
<td>0.22</td>
<td>0.24430</td>
<td>0.76</td>
<td>0.71754</td>
<td>1.90</td>
<td>0.99279</td>
</tr>
<tr>
<td>0.24</td>
<td>0.26570</td>
<td>0.80</td>
<td>0.74210</td>
<td>2.00</td>
<td>0.99532</td>
</tr>
<tr>
<td>0.26</td>
<td>0.28690</td>
<td>0.84</td>
<td>0.76514</td>
<td>2.20</td>
<td>0.99814</td>
</tr>
<tr>
<td>0.28</td>
<td>0.30788</td>
<td>0.88</td>
<td>0.78669</td>
<td>2.40</td>
<td>0.99931</td>
</tr>
<tr>
<td>0.30</td>
<td>0.32863</td>
<td>0.92</td>
<td>0.80677</td>
<td>2.60</td>
<td>0.99976</td>
</tr>
<tr>
<td>0.32</td>
<td>0.34913</td>
<td>0.96</td>
<td>0.82542</td>
<td>2.80</td>
<td>0.99992</td>
</tr>
<tr>
<td>0.34</td>
<td>0.36936</td>
<td>1.00</td>
<td>0.84270</td>
<td>3.00</td>
<td>0.99998</td>
</tr>
</tbody>
</table>
5.7 semi-infinite solid

- **Case 2 constant surface heat flux**

\[ q_s'' = q_o'' \]

\[ T(x, t) - T_i = \frac{2q_o''(\alpha t/\pi)^{1/2}}{k} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q''_o x}{k} \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \]

- **Case 3 surface convection**

\[ -k \frac{\partial T}{\partial x}\bigg|_{x=0} = h[T_\infty - T(0, t)] \]

\[ \frac{T(x, t) - T_i}{T_\infty - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \]

\[ -\left[ \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\right] \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right) \]
5.7 semi-infinite solid

- Gaussian error function 高斯误差函数 $erfw$
- Complementary error function 余误差函数 $erfcw$

$$erfcw = 1 - erfw$$
5S.2 Analytical Solution of Multidimensional Effects
Conduction in the Plane Wall, Long Cylinder and Sphere

- Analytical solutions for some simple 2-D and 3-D geometries are found in 5S.2
Example 5.6 p288

In laying water mains, utilities must be concerned with the possibility of freezing during cold periods. Although the problem of determining the temperature in soil as a function of time is complicated by changing surface conditions, reasonable estimates can be based on the assumption of a constant surface temperature over a prolonged period of cold weather. What minimum burial depth $x_m$ would you recommend to avoid freezing under conditions for which soil, initially at a uniform temperature of 20°C, is subjected to a constant surface temperature of −15°C for 60 days?

**Solution**

**Known:** Temperature imposed at the surface of soil initially at 20°C.

**Find:** The depth $x_m$ to which the soil has frozen after 60 days.

**Schematic:**

For semi-infinite solid

$$\frac{T(x_m, t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x_m}{2\sqrt{\alpha t}}\right)$$

or

$$\frac{0 - (-15)}{20 - (-15)} = 0.429 = \text{erf}\left(\frac{x_m}{2\sqrt{\alpha t}}\right)$$

Hence, from Appendix B.2

$$\frac{x_m}{2\sqrt{\alpha t}} = 0.40$$

and

$$x_m = 0.80(\alpha t)^{1/2} = 0.80(0.138 \times 10^{-6} \text{ m}^2/\text{s} \times 60 \text{ days} \times 24 \text{ h/day} \times 3600 \text{ s/h})^{1/2} = 0.68 \text{ m}$$
• 5.8 Objects with constant fluid surface temperatures or surface heat fluxes

• 5.9 Periodic heating

study by yourself
5.10 finite difference method

Finite-difference method (in 4.4, steady) \(\rightarrow\) transient problems

Control equation:

\[
\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}
\]

Central difference approximation to spatial derivatives (空间上仍然采用中心差分的形式)
5.10 finite difference method

Difference approximation to time derivative:

\[ \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \]

\[ t = p \Delta t \]

\[ \left. \frac{\partial T}{\partial t} \right|_{m,n} \approx \frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta t} \]

Forward-difference approximation

What about the space approximation?

→ Central difference at previous time.

\( T^p \): temp at previous time

\( T^{p+1} \): temp at new time
5.10 finite difference method

Explicit form of the finite-difference equation (显式)

\[
\frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p} + T_{m-1,n}^{p} - 2T_{m,n}^{p}}{\Delta x^2} + \frac{T_{m,n+1}^{p} + T_{m,n-1}^{p} - 2T_{m,n}^{p}}{\Delta y^2}
\]

\[
T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4Fo)T_{m,n}^{p}
\]

a finite-difference form of Fo number, Fo有限差分形式, 网格傅里叶数

The Explicit method advantage:

Known temp → unknown temp

Fundamentals of Heat and Mass Transfer, 6th
5.10 finite difference method

\[ \Delta x \]
Is determined by both **accuracy** and **computational requirements**

\[ \Delta t \]
Is determined by **stability requirements**

The explicit method \( \rightarrow \) **unstable solutions**
5.10 finite difference method

• Stability criterion:

The coefficient associated with the node of interest at the previous time is greater than or equal to zero.

For a 1 D interior node

\[ T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4Fo)T_{m,n}^{p} \]

\[ Fo \leq \frac{1}{2} \]

For 2D interior nodes

\[ T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p}) + (1 - 2Fo)T_{m,n}^{p} \]

\[ Fo \leq \frac{1}{4} \]

\[ Fo = \frac{\alpha \Delta t}{(\Delta x)^2} \]
5.10 finite difference method

- Energy balance method on surface nodes 1D

\[
hA(T_\infty - T_0^p) + \frac{kA}{\Delta x} (T_1^p - T_0^p) = \rho c A \frac{\Delta x}{2} \frac{T_0^{p+1} - T_0^p}{\Delta t}
\]

\[
T_0^{p+1} = 2Fo(T_1^p + Bi T_\infty) + (1 - 2Fo - 2Bi Fo)T_0^p
\]

\[\dot{E}_{in} + \dot{E}_g = \dot{E}_{st}\]

\[1 - 2Fo - 2Bi Fo \geq 0\]

\[Fo(1 + Bi) \leq \frac{1}{2}\]

Criterion for selection of time interval in 1D case

**Figure 5.12** Surface node with convection and one-dimensional transient conduction.

*Fundamentals of Heat and Mass Transfer, 6th*
**Example 5.9**

A fuel element of a nuclear reactor is in the shape of a plane wall of thickness $2L = 20$ mm and is convectively cooled at both surfaces, with $h = 1100 \text{ W/m}^2 \cdot \text{K}$ and $T_\infty = 250^\circ \text{C}$. At normal operating power, heat is generated uniformly within the element at a volumetric rate of $\dot{q}_1 = 10^7 \text{ W/m}^3$. A departure from the steady-state conditions associated with normal operation will occur if there is a change in the generation rate. Consider a sudden change to $\dot{q}_2 = 2 \times 10^7 \text{ W/m}^3$, and use the explicit finite-difference method to determine the fuel element temperature distribution after 1.5 s. The fuel element thermal properties are $k = 30 \text{ W/m} \cdot \text{K}$ and $\alpha = 5 \times 10^{-6} \text{ m}^2/\text{s}$.

**Schematic:**

[Diagram showing the fuel element with labeled parameters: $q_1 = 1 \times 10^7 \text{ W/m}^3$, $q_2 = 2 \times 10^7 \text{ W/m}^3$, $\alpha = 5 \times 10^{-6} \text{ m}^2/\text{s}$, $k = 30 \text{ W/m-K}$, $T_\infty = 250^\circ \text{C}$, $h = 1100 \text{ W/m}^2 \cdot \text{K}$, and the coolant flow.]
5.10 finite difference method

• **Implicit method** (backward-difference approximation):

\[
\frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} = \frac{T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta x)^2} \\
+ \frac{T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} - 2T_{m,n}^{p+1}}{(\Delta y)^2}
\]

\[
(1 + 2Fo)T_{m}^{p+1} - Fo (T_{m-1}^{p+1} + T_{m+1}^{p+1}) = T_{m}^{p}
\]

Unconditionally stable → no restrictions on \( \Delta t, \Delta x, \Delta y \)

无条件稳定

Solution method: Gauss-Seidel iteration, matrix inversion
Table 5.3 Transient, two-dimensional finite-difference equations ($\Delta x = \Delta y$)

(a) Explicit Method

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Finite-Difference Equation</th>
<th>Stability Criterion</th>
</tr>
</thead>
</table>
| 1. Interior node | \[
T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + (1 - 4Fo)T_{m,n}^{p})
\]
| \[Fo \leq \frac{1}{4} \] (5.76) |
| 2. Node at interior corner with convection | \[
T_{m,n}^{p+1} = Fo(2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi T_{\infty})
\]
| \[Fo(2 + Bi) \leq \frac{1}{2} \] (5.88) |
| 3. Node at plane surface with convection | \[
T_{m,n}^{p+1} = 2Fo(T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2Bi T_{\infty})
\]
| \[Fo(1 + Bi) \leq \frac{1}{4} \] (5.90) |
| 4. Node at exterior corner with convection | \[
T_{m,n}^{p+1} = Fo(T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2Bi T_{\infty})
\]
| \[Fo(1 + Bi) \leq \frac{1}{4} \] (5.90) |

(b) Implicit Method

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Finite-Difference Equation</th>
<th>Stability Criterion</th>
</tr>
</thead>
</table>
| 1. Interior node | \[
(1 + 4Fo)T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1} + (1 - 4Fo)T_{m,n}^{p+1})
\]
| \[Fo \leq \frac{1}{4} \] (5.80) |
| 2. Node at interior corner with convection | \[
(1 + 4Fo(1 + \frac{1}{3}Bi))T_{m,n}^{p+1} - \frac{2}{3}Fo \cdot (T_{m+1,n}^{p+1} + 2T_{m-1,n}^{p+1} + 2T_{m,n+1}^{p+1} + 2T_{m,n-1}^{p+1} + T_{m,n}^{p+1})
\]
| \[Fo(3 + Bi) \leq \frac{3}{4} \] (5.86) |
| 3. Node at plane surface with convection | \[
(1 + 2Fo(2 + Bi))T_{m,n}^{p+1} - Fo(2T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1})
\]
| \[Fo(2 + Bi) \leq \frac{1}{2} \] (5.88) |
| 4. Node at exterior corner with convection | \[
(1 + 4Fo(1 + Bi))T_{m,n}^{p+1} - 2Fo(T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1})
\]
| \[Fo(1 + Bi) \leq \frac{1}{4} \] (5.90) |

*To obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set $Bi$ equal to zero.
**Example 5.10**

**implicit method**

A thick slab of copper initially at a uniform temperature of 20°C is suddenly exposed to radiation at one surface such that the net heat flux is maintained at a constant value of $3 \times 10^5$ W/m². Using the explicit and implicit finite-difference techniques with a space increment of $\Delta x = 75$ mm, determine the temperature at the irradiated surface and at an interior point that is 150 mm from the surface after 2 min have elapsed. Compare the results with those obtained from an appropriate analytical solution.

**Solution**

**Known:** Thick slab of copper, initially at a uniform temperature, is subjected to a constant net heat flux at one surface.

**Find:**

1. Using the explicit finite-difference method, determine temperatures at the surface and 150 mm from the surface after an elapsed time of 2 min.
2. Repeat the calculations using the implicit finite-difference method.
3. Determine the same temperatures analytically.

**Schematic:**

![Schematic diagram showing heat flux and finite-difference approach]

$q''_0 = 3 \times 10^5$ W/m²

$\Delta x = 75$ mm

$\Delta x/2$

$q''_o$

$q''_{\text{cond}}$

$m - 1$

$m$

$m + 1$
5.11 summary

- Lumped capacitance method
- Dimensionless number : Bi, Fo
- Analytical method (exact and approximate solutions)
- Numerical method (explicit and implicit)
Homework Assignment

Lumped capacitance method
- 5.7
- 5.12
- 5.16

Numerical method
- 5.121(a)
- 5.123